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Contact superconformal algebras and their representations

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1.- A *superconformal algebra* is a complex $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G} = \oplus_i \mathcal{G}_i$, such that
1) $\mathcal{G}$ is simple,
2) $\mathcal{G}$ contains the centerless Virasoro algebra i.e. the Lie algebra with the basis L_n ($n \in \mathbb{Z}$) and commutation relations $[L_m, L_n] = (m-n)L_{m+n}$ as a subalgebra,
3) $\mathcal{G}_i = \{x \in \mathcal{G} \mid [L_0, x] = ix\}$ and dim $\mathcal{G}_i < C$, where C is a constant independent of i [1, 2].

To describe superconformal algebras, consider the Grassmann algebra $\Lambda(N)$ in N variables $\theta_1, \ldots, \theta_N$. Let $\Lambda(1,N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$ be an associative superalgebra with natural multiplication and with the following parity of generators: $p(t) = \bar{0}$, $p(\theta_i) = \bar{1}$ for $i = 1, \ldots, N$. Let $W(N)$ be the Lie superalgebra of all derivations of $\Lambda(1,N)$. Every $D \in W(N)$ is represented by a differential operator,

$$D = f\partial_t + \sum_{i=1}^{N} f_i \partial_i, \text{ where } f, f_i \in \Lambda(1,N). \tag{1}$$

The main examples of superconformal algebras are the following three series: the series $W(N)$ ($N \geq 0$), the series $S'(N, \alpha)$ ($N \geq 2$) of one-parameter families of deformations of the divergence-free subalgebra of $W(N)$, and the series of *contact superalgebras* $K'(N)$ ($N \geq 0$)([1, 2]). By definition,

$$K(N) = \{D \in W(N) \mid D\Omega = f\Omega \text{ for some } f \in \Lambda(1,N)\}, \tag{2}$$

where $\Omega = dt - \sum_{i=1}^{N} \theta_i d\theta_i$ is a differential 1-form, which is called a *contact form* [1, 2, 3]. There is one-to one correspondence between the differential operators $D \in K(N)$ and the functions $f \in \Lambda(1,N)$. The correspondence $f \leftrightarrow D_f$ is given by

$$D_f = \Delta(f)\partial_t + \partial_t(f) \sum_{i=1}^{N} \theta_i \partial_i + (-1)^{p(f)} \sum_{i=1}^{N} \partial_i(f)\partial_i, \tag{3}$$

where $\Delta(f) = 2f - \sum_{i=1}^{N} \theta_i \partial_i(f)$. The Lie bracket in $K(N)$ is identified with the contact bracket in $\Lambda(1,N)$:

$$\{f, g\} = \Delta(f)\partial_t(g) - \partial_t(f)\Delta(g) + (-1)^{p(f)} \sum_{i=1}^{N} \partial_i(f)\partial_i(g), \tag{4}$$

so that $[D_f, D_g] = D_{\{f,g\}}$. The superalgebras $K(N)$ are known to the physicists as the $SO(N)$ *superconformal algebras* [4]. They are simple, except when $N = 4$.

If $N = 4$, then the derived superalgebra $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one defined from the exact sequence

$$0 \to K'(4) \to K(4) \to \mathbb{C}D_{t^{-1}\theta_1\theta_2\theta_3\theta_4} \to 0. \quad (5)$$

There exists no nontrivial 2-cocycles on $K'(N)$ if $N > 4$. If $N \leq 3$, then there exists, up to equivalence, one nontrivial 2-cocycle [1]. Let $\hat{K}'(N)$ be the corresponding central extension of $K'(N)$. $\hat{K}'(1)$ is isomorphic to the *Neveu-Schwarz algebra*, and $\hat{K}'(2)$ is isomorphic to the so-called $N = 2$ *superconformal algebra* (see references in [8]). The superalgebra $K'(4)$ has 3 independent central extensions [1]. Note that $K'(4)$ is spanned by 16 field; it is the largest among superconformal algebras admitting central extensions and, therefore is one of the most interesting ones.

The superalgebra $K'(N)$, where $N \geq 0$, has a two-parameter family of representations in the superspace spanned by 2^N fields. In fact, $K'(N)$ acts in a natural way on the superspace of "densities" of the form $t^\alpha g \Omega^\beta$, where $g \in \Lambda(1, N)$, and α and β are fixed complex numbers [1]:

$$D_f(t^\alpha g\Omega^\beta) = (D_f(t^\alpha g) + (-1)^{p(f)p(g)} 2\beta t^\alpha g \partial_t(f))\Omega^\beta. \quad (6)$$

The superalgebra $K'(4)$ has in addition a one-parameter family of spinor-like tiny irreducible representations realized on just 4 fields instead of the usual 16. The construction of this representation is based on the embedding of a nontrivial central extension of $K'(4)$ into the Lie superalgebra of pseudodifferential symbols on the supercircle $S^{1|2}$ [8]. Note that there is no analogous representation of $K'(2N)$ realized on 2^N fields for $N \geq 3$.

2.- The *Poisson algebra P of pseudodifferential symbols on the circle* is formed by the formal series $A(t, \xi) = \sum_{-\infty}^{n} a_i(t)\xi^i$, where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the variable ξ corresponds to ∂_t. The Poisson bracket is defined as follows:

$$\{A(t, \xi), B(t, \xi)\} = \partial_\xi A(t, \xi)\partial_t B(t, \xi) - \partial_t A(t, \xi)\partial_\xi B(t, \xi). \quad (7)$$

The Poisson algebra P has a well-known deformation P_h, where $h \in]0, 1]$. The associative multiplication in the vector space P is determined as follows:

$$A(t, \xi) \circ_h B(t, \xi) = \sum_{n \geq 0} \frac{h^n}{n!} \partial_\xi^n A(t, \xi)\partial_t^n B(t, \xi). \quad (8)$$

The Lie algebra structure on P_h is given by $[A, B]_h = A \circ_h B - B \circ_h A$, so that the family P_h contracts to P [5]. $P_{h=1}$ is called the *Lie algebra of pseudodifferential symbols on the circle*.

Let $\Theta(N)$ be the Grassman algebra with generators $\theta_1, \ldots, \theta_N, \bar{\theta}_1, \ldots, \bar{\theta}_N$, where $\bar{\theta}_i = \partial_i$ for $i = 1, \ldots, N$. The *Poisson superalgebra $P(N)$ of pseudodifferential symbols on the supercircle* $S^{1|N}$ has the underlying vector space $P \otimes \Theta(N)$. The Poisson bracket is defined as follows:

$$\{A, B\} = \partial_\xi A \partial_t B - \partial_t A \partial_\xi B - (-1)^{p(A)}(\sum_{i=1}^{N} \partial_{\theta_i} A \partial_{\bar{\theta}_i} B + \partial_{\bar{\theta}_i} A \partial_{\theta_i} B). \quad (9)$$

Let $\Theta_h(N)$ be an associative superalgebra with generators $\theta_1,\dots,\theta_N,\partial_1,\dots,\partial_N$ and relations: $\theta_i\theta_j = -\theta_j\theta_i, \partial_i\partial_j = -\partial_j\partial_i, \partial_i\theta_j = h\delta_{i,j} - \theta_j\partial_i$. Let $P_h(N) = P_h \otimes \Theta_h(N)$ be an associative superalgebra with the product

$$(A\otimes X)(B\otimes Y) = \frac{1}{h}(A\circ_h B)\otimes(XY), \tag{10}$$

where $A, B \in P_h$, and $X, Y \in \Theta_h(N)$. Correspondingly, the Lie bracket in $P_h(N)$ is $[A,B]_h = AB - (-1)^{p(A)p(B)}BA$, and $\lim_{h\to 0}[A,B]_h = \{A,B\}$ for $A, B \in P_h(N)$. There exist natural embeddings: $W(N) \subset P(N)$ and $W(N) \subset P_h(N)$. $P_{h=1}(N)$ is the *Lie superalgebra of pseudodifferential symbols on* $S^{1|N}$.

The Lie algebra $P_{h=1}$ has two independent central extensions [5]. Analogously, there exist, up to equivalence, two nontrivial 2-cocycles, c_ξ and c_t, on $P_{h=1}(N)$ [8]. Let $x = \xi, t$ and $\hat{x} = t, \xi$, respectively, and let $\log x$ be the derivation of the ring $P_{h=1}$ defined by

$$[\log x, A(t,\xi)] = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\partial_{\hat{x}}^k A(t,\xi)x^{-k}, \tag{11}$$

(see [5]). Then $c_x(A\otimes X, B\otimes Y) =$ the coefficient of $t^{-1}\xi^{-1}\theta_1\dots\theta_N\partial_1\dots\partial_N$ in the expression for $([\log x, A]\circ_{h=1} B)\otimes(XY)$, where $A, B\in P_{h=1}$, and $X, Y \in \Theta_{h=1}(N)$.

There exists an embedding for $N \geq 0$:

$$K(2N) \subset P(N). \tag{12}$$

To explain this embedding, note that in general, a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra [6]. In particular, the Lie algebra $Vect(S^1)$ of smooth vector fields on the circle can be embedded into the Poisson algebra of functions on the cylinder $\dot{T}^*S^1 = T^*S^1\setminus S^1$ with the removed zero section. One can introduce the Darboux coordinates $(q,p) = (t,\xi)$ on this manifold. The symbols of differential operators are functions on $\dot{T}^*S^1$ which are formal Laurent series in p with coefficients periodic in q. Correspondingly, they define Hamiltonian vector fields on $\dot{T}^*S^1$ [5]. Then $Vect(S^1)$ is realized as the subalgebra of the Lie algebra of Hamiltonian vector fields, consisting of the fields with Hamiltonians which are homogeneous of degree 1. This condition holds in general, if one considers the *symplectification* of a contact manifold [6], and it can be generalized to the supercase.

Let $P(N) = \oplus_{j\in\mathbb{Z}}P^j(N)$ be the $\mathbb{Z}$-grading of the associative superalgebra $P(N)$ defined by $\deg\xi = \deg\bar{\theta}_i = 1, \deg t = \deg\theta_i = 0$ for $i = 1,\dots,N$. With respect to the Poisson bracket, $\{P^j(N), P^k(N)\}\subset P^{j+k-1}(N)$. Then $P^1(N)$ is a subalgebra of $P(N)$, and it is isomorphic to $K(2N)$ [8].

3.- A natural question is whether there exists an embedding of $K(2N)$ into $P_h(N)$. If $N=1$, then $P^1(1) = W(1)\cong K(2)$. Thus, clearly, $K(2)\subset P_h(1)$. If $N=2$, then $K'(4)\subset P(2)$ is defined from the exact sequence

$$0\to K'(4)\to P^1(2)\to \mathbb{C}t^{-1}\xi^{-1}\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2\to 0. \tag{13}$$

The 2-cocycles on $K'(4)$ in this realization are defined as follows. Let

$$\begin{aligned} c(t^n\xi, t^k\xi^{-1}\theta_1\theta_2\bar\theta_1\bar\theta_2) &= A\delta_{n+k,0}, n\neq 1,\\ c(t^n\xi\theta_i, t^k\xi^{-1}\theta_j\bar\theta_1\bar\theta_2) &= A(-1)^j\delta_{n+k,0}, i\neq j,\\ c(t^n\xi\theta_1\theta_2, t^k\xi^{-1}\bar\theta_1\bar\theta_2) &= A\delta_{n+k,0},\end{aligned} \tag{14}$$

$$\begin{aligned} c(t^n\bar\theta_i, t^k\theta_1\theta_2\bar\theta_j) &= B(-1)^j\delta_{n+k,0}, i\neq j,\\ c(t^n\theta_1\bar\theta_i, t^k\theta_2\bar\theta_j) &= B(-1)^i\delta_{n+k,0}, i\neq j,\end{aligned}$$

$$\begin{aligned} c(t^n\xi^{-1}\theta_1\theta_2\bar\theta_1\bar\theta_2, t^k\theta_i\bar\theta_i) &= C\delta_{n+k+1,0}, n\neq -1,\\ c(t^n\xi^{-1}\theta_i\bar\theta_1\bar\theta_2, t^k\theta_1\theta_2\bar\theta_i) &= -C\delta_{n+k+1,0},\\ c(t^n\xi^{-1}\theta_1\theta_2\bar\theta_1\bar\theta_2, t^k\xi^{-1}\theta_1\theta_2\bar\theta_1\bar\theta_2) &= -\frac{2C}{n+1}\delta_{n+k+2,0}, n\neq -1,\end{aligned}$$

where $A, B, C \in \mathbb{Z}$. Then three linearly independent 2-cocycles c_1, c_2, and c_3 on $K'(4)$ are given by

$$\begin{aligned} c_1: &\quad A=1, B=0, C=0,\\ c_2: &\quad A=n, B=n, C=0,\\ c_3: &\quad A=n, B=0, C=1.\end{aligned} \tag{15}$$

Note that these cocycles are represented by linear functions of the indexes. In [1] and [3] another formulas for cocycles are given, which involve the usual "cubic term" of the Virasoro cocycle. Note that the exterior derivations of $K'(4)$ are $Der_{ext}K'(4) = \mathbb{C}D$, where D is the derivation defined by (13): $D(x) = \{t^{-1}\xi^{-1}\theta_1\theta_2\bar\theta_1\bar\theta_2, x\}$ for $x \in K'(4)$. There is a natural action of $Der_{ext}K'(4)$ on the cohomology space $H^2(K'(4), \mathbb{C})$. Under this action $D(c_1) = D(c_2) = 0, D(c_3) = -2c_1$. Correspondingly, the exterior automorphism $ExpD$ of $K'(4)$ associates the cocycles c_3 and $c_3 - 2c_1$. Thus, up to automorphisms, there are two independent 2-cocycles on $K'(4)$: c_1 and c_2 (cf. [7]).

The embedding $K'(4) \subset P(2)$ can be obtained as follows. Consider the one-parameter family $S(2, \alpha)$ ($\alpha \in \mathbb{C}$) of deformations of the Lie superalgebra of divergence-free derivations of $\Lambda(1, 2)$ [1]. The exterior derivations of $S'(2, \alpha)$ form an $\mathfrak{sl}(2)$ if $\alpha \in \mathbb{Z}$. The exterior derivations of $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ generate a subalgebra of $P(2)$ isomorphic to the loop algebra $\tilde{\mathfrak{sl}}(2)$ ($\mathfrak{sl}(2)$ corresponds to $\alpha = 1$). The family $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ and $\tilde{\mathfrak{sl}}(2)$ generate a Lie superalgebra isomorphic to $K'(4)$. The similar construction for each $h \in]0, 1]$ gives an embedding of a nontrivial central extension of $K'(4)$

$$\hat{K}'(4) \subset P_h(2) \tag{16}$$

such that the central element is $h \in P_h(2)$, and $\lim_{h\to 0}\hat{K}'(4) = K'(4) \subset P(2)$ [8]. The superalgebra $K'(4) \subset P(2)$ is spanned by $W(2)$ together with four fields F^i_n, where $i = 0, 1, 2, 3$, and $n \in \mathbb{Z}$:

$$\begin{aligned} F^0_n &= t^n\xi^{-1}\bar\theta_1\bar\theta_2,\\ F^i_n &= t^n\xi^{-1}\theta_i\bar\theta_1\bar\theta_2, \quad i=1,2,\\ F^3_n &= t^n\xi^{-1}\theta_1\theta_2\bar\theta_1\bar\theta_2, \quad n\neq -1.\end{aligned} \tag{17}$$

The superalgebra $\hat{K}'(4) \subset P_h(2)$ is spanned by $W(2)$ together with four fields $F^i_{n,h}$:

$$\begin{aligned} F^0_{n,h} &= (\xi^{-1} \circ_h t^n)\partial_1\partial_2, \\ F^i_{n,h} &= (\xi^{-1} \circ_h t^n)\partial_1\partial_2\theta_i, \quad i = 1,2, \\ F^3_{n,h} &= (\xi^{-1} \circ_h t^n)\partial_1\partial_2\theta_1\theta_2 + \frac{h}{n+1}t^{n+1}, \quad n \neq -1, \end{aligned} \tag{18}$$

and the central element h. The corresponding 2-cocycle on $K'(4)$ is c_1. Note that the cocycles c_1 and c_2 are equal, respectively, to the restrictions of the cocycles c_t and c_ξ on $P_{h=1}(2)$.

The embedding (16) in the case where $h = 1$ allows to obtain a one-parameter family of irreducible representations of $\hat{K}'(4)$ in the superspace spanned by 2 even fields and 2 odd fields where the value of the central charge is equal to one [8]. To describe this representation, consider the superspace V^μ spanned by the functions $t^\mu g$, where $g \in \Lambda(1,2)$, and $\mu \in \mathbb{R} \setminus \mathbb{Z}$. Let $\{v^i_m\}$, where $m \in \mathbb{Z}$ and $i = 0,1,2,3$, be the following basis in V^μ:

$$v^0_m = \frac{1}{m+\mu}t^{m+\mu}, \quad v^1_m = t^{m+\mu}\theta_1, \quad v^2_m = t^{m+\mu}\theta_2, \quad v^3_m = t^{m+\mu}\theta_1\theta_2. \tag{19}$$

Every $D \in W(2)$ is a derivation of V^μ. To define an action of $F^i_{n,1}$ on V^μ, one can interpret ξ^{-1} as the anti-derivative on the space of functions $t^\mu\mathbb{C}[t,t^{-1}]$. Then

$$\begin{aligned} F^0_{n,1}(v^3_m) &= -v^0_{m+n+1}, \\ F^1_{n,1}(v^2_m) &= -v^0_{m+n+1}, \quad F^2_{n,1}(v^1_m) = v^0_{m+n+1}, \\ F^3_{n,1}(v^i_m) &= \frac{1}{n+1}v^i_{m+n+1}, \quad i = 0,1,2,3; \quad n \neq -1. \end{aligned} \tag{20}$$

The central element in $\hat{K}'(4)$ is 1, and it acts by the identity operator. In this way one obtains a family of representations of $\hat{K}'(4)$ in the superspace spanned by four fields v^i_m, where $i = 0,1,2,3$, and μ appears as an arbitrary complex parameter [8].

No analogous embedding of $K(2N)$ into $P_h(N)$ exists if $N \geq 3$.

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